A class of positive atomic maps

2008 J. Phys. A: Math. Theor. 41 215201

(http://iopscience.iop.org/1751-8121/41/21/215201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 215201 (13pp) doi:10.1088/1751-8113/41/21/215201

A class of positive atomic maps

Dariusz Chruściński and Andrzej Kossakowski

Institute of Physics, Nicolaus Copernicus University, Grudziądzka 5/7, 87-100 Toruń, Poland

Received 29 November 2007, in final form 25 March 2008
Published 7 May 2008
Online at stacks.iop.org/JPhysA/41/215201

Abstract
We construct a new class of positive indecomposable maps in the algebra of $d \times d$ complex matrices. These maps are characterized by the 'weakest' positivity property and for this reason they are called atomic. This class provides a new rich family of atomic entanglement witnesses which define an important tool for investigating quantum entanglement. It turns out that they are able to detect states with the 'weakest' quantum entanglement.

PACS numbers: 03.67.Mn, 03.65.Ud, 02.30.Tb

1. Introduction

One of the most important problems of quantum information theory [1] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e. whether it is separable or entangled. For low-dimensional systems there exists a simple necessary and sufficient condition for separability. The celebrated Peres–Horodecki criterion [2, 3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transpose is positive (one calls it a PPT state). Unfortunately, for higher-dimensional systems there is no single universal separability condition.

The most general approach to the separability problem is based on the following observation [4]: a state ρ of a bipartite system living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff $\mathrm{Tr}(W\rho) \geqslant 0$ for any Hermitian operator W satisfying $\mathrm{Tr}(W P_A \otimes P_B) \geqslant 0$, where P_A and P_B are projectors acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Recall, that a Hermitian operator $W \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is an entanglement witness [4, 5] iff: (i) it is not positively defined, i.e. $W \ngeqslant 0$, and (ii) $\mathrm{Tr}(W\sigma) \geqslant 0$ for all separable states σ. A bipartite state ρ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff there exists an entanglement witness W detecting ρ, i.e. such that $\mathrm{Tr}(W\rho) < 0$. Clearly, the construction of entanglement witnesses is a hard task. It is easy to construct W which is not positive, i.e. has at least one negative eigenvalue, but it is very difficult to check that $\mathrm{Tr}(W\sigma) \geqslant 0$ for all separable states σ.

The separability problem may be equivalently formulated in terms of positive maps [4]: a state ρ is separable iff $(\mathbb{1} \otimes \Lambda)\rho$ is positive for any positive map Λ which sends positive operators on $\mathcal{H}_B$ into positive operators on $\mathcal{H}_A$. Due to the celebrated Choi–Jamiołkowski

[6, 7] isomorphism there is a one to one correspondence between entanglement witnesses and positive maps which are not completely positive: if Λ is such a map, then $W_\Lambda := (\mathbb{1} \otimes \Lambda)P^+$ is the corresponding entanglement witness (P^+ stands for the projector onto the maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$). Unfortunately, in spite of the considerable effort, the structure of positive maps (and hence also the set of entanglement witnesses) is rather poorly understood [7–44].

Now, among positive linear maps the crucial role is played by indecomposable maps. These are maps which may detect entangled PPT states. Among indecomposable maps there is a set of maps which are characterized by the 'weakest positivity' property: they are called *atomic maps* and they may be used to detect states with the 'weakest' entanglement. The corresponding entanglement witnesses we call indecomposable and atomic, respectively.

There are only a few examples of indecomposable maps in the literature (for the list see, e.g. the recent paper [44]). The set of atomic ones is considerably smaller. Interestingly, Choi's first example [7] of an indecomposable positive map turned out to be an atomic one. Recently, Hall [45] and Breuer [46] considered a new family of indecomposable maps (they were applied by Breuer [47] in the study of rotationally invariant bipartite states, see also [48]). In this paper we show that these maps are not only indecomposable but also atomic. Moreover, we show how to generalize this family to obtain a large family of new positive maps. We study which maps within this family are indecomposable and which are atomic.

The paper is organized as follows: in the following section we introduce a natural hierarchy of positive convex cones in the space of (unnormalized) states of bipartite $d \otimes d$ quantum systems and recall basic notions from the theory of entanglement witnesses and positive maps. Section 3 discusses properties of the recently introduced indecomposable maps [45, 46] and provides the proof that these maps are atomic. Finally, section 4 introduces a new class of indecomposable maps and studies which maps within this class are atomic. A brief discussion is included in the last section.

2. Quantum entanglement versus positive maps

Let M_d denote a set of $d \times d$ complex matrices and let M_d^+ be a convex set of semi-positive elements in M_d, that is, M_d^+ defines a space of (unnormalized) states of d-level quantum system. Let us recall [50] that for any normalized positive operator ρ on $\mathcal{H} \otimes \mathcal{H}$ one may define its Schmidt number

$$\mathrm{SN}(\rho) = \min_{p_k, \psi_k} \{\max_k \mathrm{SR}(\psi_k)\}, \tag{2.1}$$

where the minimum is taken over all possible pure states decompositions

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \tag{2.2}$$

with $p_k \geqslant 0$, $\sum_k p_k = 1$ and ψ_k are normalized vectors in $\mathcal{H} \otimes \mathcal{H}$. The Schmidt rank $\mathrm{SR}(\psi)$ denotes the number of non-vanishing Schmidt coefficients in the Schmidt decomposition of ψ. This number characterizes the minimum Schmidt rank of the pure states that are needed to construct such density matrix. It is evident that $1 \leqslant \mathrm{SN}(\rho) \leqslant d = \dim \mathcal{H}$. Moreover, ρ is separable iff $\mathrm{SN}(\rho) = 1$. This notion enables one to introduce the following family of positive cones:

$$V_r = \{\rho \in (M_d \otimes M_d)^+ | \mathrm{SN}(\rho) \leqslant r\}. \tag{2.3}$$

One has the following chain of inclusions:

$$V_1 \subset \cdots \subset V_d \equiv (M_d \otimes M_d)^+. \tag{2.4}$$

Clearly, V_1 is a cone of separable (unnormalized) states and $V_d \setminus V_1$ stands for a set of entangled states. Note, that a partial transposition $(\mathbb{1}_d \otimes \tau)$ gives rise to another family of cones:

$$V^l = (\mathbb{1}_d \otimes \tau)V_l, \tag{2.5}$$

such that $V^1 \subset \cdots \subset V^d$. One has $V_1 = V^1$, together with the following hierarchy of inclusions:

$$V_1 = V_1 \cap V^1 \subset V_2 \cap V^2 \subset \cdots \subset V_d \cap V^d. \tag{2.6}$$

Note, that $V_d \cap V^d$ is a convex set of PPT (unnormalized) states. Finally, $V_r \cap V^s$ is a convex subset of PPT states ρ such that $\mathrm{SN}(\rho) \leqslant r$ and $\mathrm{SN}[(\mathbb{1}_d \otimes \tau)\rho] \leqslant s$.

Consider now a set of positive maps $\varphi : M_d \longrightarrow M_d$, i.e. maps such that $\varphi(M_d^+) \subseteq M_d^+$. Following Størmer definition [9], a positive map φ is k-positive iff

$$(\mathbb{1} \otimes \varphi)(V_k) \subset (M_d \otimes M_d)^+, \tag{2.7}$$

and it is k-copositive iff

$$(\mathbb{1} \otimes \varphi)(V^k) \subset (M_d \otimes M_d)^+. \tag{2.8}$$

Denoting by P_k (P^k) a convex cone of k-positive (k-copositive) maps one has the following chains of inclusions:

$$P_d \subset P_{d-1} \subset \cdots \subset P_2 \subset P_1, \tag{2.9}$$

and

$$P^d \subset P^{d-1} \subset \cdots \subset P^2 \subset P^1, \tag{2.10}$$

where P_d (P^d) stands for a set of completely positive (copositive) maps.

A positive map $\varphi : M_d \longrightarrow M_d$ is *decomposable* iff $\varphi \in P_d + P^d$, that is, φ can be written as $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \in P_d$ and $\varphi_2 \in P^d$. Otherwise φ is *indecomposable*. Indecomposable maps can detect entangled states from $V_d \cap V^d \equiv$ PPT, that is, bound entangled states. Finally, a positive map is *atomic* iff $\varphi \notin P_2 + P^2$. The importance of atomic maps follows from the fact that they may be used to detect the 'weakest' bound entanglement, that is, atomic maps can detect states from $V_2 \cap V^2$.

Actually, Størmer definition [9] is rather difficult to apply in practice. Using the Choi–Jamiołkowski isomorphism [6, 7] we may assign to any linear map $\varphi : M_d \rightarrow M_d$ the following operator $\widehat{\varphi} \in M_d \otimes M_d$:

$$\widehat{\varphi} = (\mathbb{1}_d \otimes \varphi)P^+ \in M_d \otimes M_d, \tag{2.11}$$

where P^+ stands for (unnormalized) maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$. If $e_i\,(i = 1, \ldots, d)$ is an orthonormal basis in $\mathbb{C}^d$, then

$$\widehat{\varphi} = \sum_{i,j=1}^{d} e_{ij} \otimes \varphi(e_{ij}), \tag{2.12}$$

where $e_{ij} = |i\rangle\langle j|$ defines a basis in M_d. It is clear that if φ is a positive but not completely positive map then the corresponding operator $\widehat{\varphi}$ is an entanglement witness. Now, the space of linear maps $\mathcal{L}(M_d, M_d)$ is endowed with a natural inner product:

$$(\varphi, \psi) = \mathrm{Tr}\left(\sum_{\alpha=1}^{d^2} \varphi(f_\alpha)^* \psi(f_\alpha)\right), \tag{2.13}$$

where f_α is an arbitrary orthonormal basis in M_d. Taking $f_\alpha = e_{ij}$, one finds

$$(\varphi, \psi) = \mathrm{Tr}\left(\sum_{i,j=1}^{d} \varphi(e_{ij})^* \psi(e_{ij})\right) = \mathrm{Tr}\left(\sum_{i,j=1}^{d} \varphi(e_{ij}) \psi(e_{ji})\right). \tag{2.14}$$

The above defined inner product is compatible with the standard Hilbert–Schmidt product in $M_d \otimes M_d$. Indeed, taking $\widehat{\varphi}$ and $\widehat{\psi}$ corresponding to φ and ψ, one has

$$(\widehat{\varphi}, \widehat{\psi})_{\mathrm{HS}} = \mathrm{Tr}(\widehat{\varphi}^* \widehat{\psi}) \tag{2.15}$$

and using (2.12) one easily finds

$$(\varphi, \psi) = (\widehat{\varphi}, \widehat{\psi})_{\mathrm{HS}}, \tag{2.16}$$

that is, formula (2.12) defines an inner product isomorphism. This way one establishes the duality between maps from $\mathcal{L}(M_d, M_d)$ and operators from $M_d \otimes M_d$ [32]: for any $\rho \in M_d \otimes M_d$ and $\varphi \in \mathcal{L}(M_d, M_d)$ one defines

$$\langle \rho, \varphi \rangle := (\rho, \widehat{\varphi})_{\mathrm{HS}}. \tag{2.17}$$

In the space of entanglement witnesses $\mathbf{W}$ one may introduce the following family of subsets $\mathbf{W}_r \subset M_d \otimes M_d$:

$$\mathbf{W}_r = \{W \in M_d \otimes M_d | \mathrm{Tr}(W\rho) \geqslant 0, \rho \in V_r\}. \tag{2.18}$$

One has

$$(M_d \otimes M_d)^+ \equiv \mathbf{W}_d \subset \cdots \subset \mathbf{W}_1. \tag{2.19}$$

Clearly, $\mathbf{W} = \mathbf{W}_1 \smallsetminus \mathbf{W}_d$. Moreover, for any $k > 1$, entanglement witnesses from $\mathbf{W}_l \smallsetminus \mathbf{W}_k$ can detect entangled states from $V_k \smallsetminus V_l$, i.e. states ρ with Schmidt number $l < \mathrm{SN}(\rho) \leqslant k$. In particular $W \in \mathbf{W}_k \smallsetminus \mathbf{W}_{k+1}$ can detect state ρ with $\mathrm{SN}(\rho) = k$.

Consider now the following class:

$$\mathbf{W}_r^s = \mathbf{W}_r + (\mathbb{1} \otimes \tau)\mathbf{W}_s, \tag{2.20}$$

that is, $W \in \mathbf{W}_r^s$ iff

$$W = P + (\mathbb{1} \otimes \tau) Q, \tag{2.21}$$

with $P \in \mathbf{W}_r$ and $Q \in \mathbf{W}_s$. Note, that $\mathrm{Tr}(W\rho) \geqslant 0$ for all $\rho \in V_r \cap V^s$. Hence such W can detect PPT states ρ such that $\mathrm{SN}(\rho) \geqslant r$ and $\mathrm{SN}[(\mathbb{1}_d \otimes \tau)\rho] \geqslant s$. Entanglement witnesses from $\mathbf{W}_d^d$ are called decomposable [49]. They cannot detect PPT states. One has the following chain of inclusions:

$$\mathbf{W}_d^d \subset \cdots \subset \mathbf{W}_2^2 \subset \mathbf{W}_1^1 \equiv \mathbf{W}. \tag{2.22}$$

The 'weakest' entanglement can be detected by elements from $\mathbf{W}_1^1 \smallsetminus \mathbf{W}_2^2$. We shall call them *atomic entanglement witnesses*. It is clear that W is an atomic entanglement witness if there is an entangled state $\rho \in V_2 \cap V^2$ such that $\mathrm{Tr}(W\rho) < 0$. The knowledge of atomic witnesses, or equivalently atomic maps, is crucial: knowing this set we would be able to distinguish all entangled states from separable ones.

3. A class of atomic maps of Breuer and Hall

Recently Breuer and Hall [46, 45] analyzed the following class of positive maps $\varphi : M_d \longrightarrow M_d$

$$\varphi_U^d(X) = \mathrm{Tr}(X)\mathbb{I}_d - X - UX^T U^*, \tag{3.1}$$

where U is an antisymmetric unitary matrix in $\mathbb{C}^d$ which implies that d is necessarily even and $d \geqslant 4$ (for $d = 2$ the above map is trivial $\varphi_U^d(X) = 0$). One may easily add a normalization factor such that

$$\widetilde{\varphi}_U^d = \frac{1}{d-2}\varphi_U^d, \tag{3.2}$$

is unital, that is, $\widetilde{\varphi}^d_U(\mathbb{I}_d) = \mathbb{I}_d$. The characteristic feature of these maps is that for any rank one projector P its image under φ^d_U reads as follows:

$$\varphi^d_U(P) = \mathbb{I}_d - P - Q, \tag{3.3}$$

where Q is again rank one projector satisfying $PQ = 0$. Hence $\varphi^d_U(P) \geqslant 0$ which proves positivity of φ^d_U. It was shown [45, 46] that these maps are not only positive but also indecomposable.

Interestingly, maps considered by Breuer and Hall are closely related to a positive map introduced long ago by Robertson [17–20]. The Robertson map $\varphi_R : M_4 \longrightarrow M_4$ is defined as follows:

$$\varphi_R\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22}\end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} \mathbb{I}_2 \operatorname{Tr} X_{22} & X_{12} + R(X_{21}) \\ \hline X_{21} + R(X_{12}) & \mathbb{I}_2 \operatorname{Tr} X_{11}\end{array}\right), \tag{3.4}$$

where $X_{kl} \in M_2$ and $R : M_2 \longrightarrow M_2$ is defined by

$$R(a) = \mathbb{I}_2 \operatorname{Tr} a - a, \tag{3.5}$$

that is, R is nothing but the reduction map. Introducing an orthonormal basis $(e_1, \ldots, e_4)$ in $\mathbb{C}^4$ and defining $e_{ij} = |e_i\rangle\langle e_j|$, one easily finds the following formulae:

$$\begin{aligned}
\varphi_R(e_{11}) &= \varphi_4(e_{22}) = \tfrac{1}{2}(e_{33} + e_{44}),\\
\varphi_R(e_{33}) &= \varphi_4(e_{44}) = \tfrac{1}{2}(e_{11} + e_{22}),\\
\varphi_R(e_{13}) &= \tfrac{1}{2}(e_{13} + e_{42}),\\
\varphi_R(e_{14}) &= \tfrac{1}{2}(e_{14} - e_{32}),\\
\varphi_R(e_{23}) &= \tfrac{1}{2}(e_{23} - e_{41}),\\
\varphi_R(e_{24}) &= \tfrac{1}{2}(e_{24} + e_{31}),\\
\varphi_R(e_{12}) &= \varphi_R(e_{34}) = 0.
\end{aligned} \tag{3.6}$$

Note, that the Robertson map is unital, i.e. $\varphi_R(\mathbb{I}_4) = \mathbb{I}_4$.

Theorem 1. *The normalized Breuer–Hall map $\widetilde{\varphi}^4_U$ in $d = 4$ is unitary equivalent to the Robertson map φ_R, that is*

$$\widetilde{\varphi}^4_U(X) = U_1\varphi_R(U_2^* X U_2)U_1^*, \tag{3.7}$$

for some unitaries U_1 and U_2.

Proof. Let us observe that

$$\Gamma\varphi_R(X)\Gamma^* = \widetilde{\varphi}^4_{U_0}(X), \tag{3.8}$$

where Γ is the following 4×4 unitary matrix

$$\Gamma = \left(\begin{array}{c|c} \mathbb{I}_2 & 0 \\ \hline 0 & -\mathbb{I}_2\end{array}\right), \tag{3.9}$$

and $\widetilde{\varphi}^4_{U_0}$ is a normalized Breuer–Hall map (3.1) corresponding to 4×4 antisymmetric unitary matrix[1]

$$U_0 = \mathrm{i}\mathbb{I}_2 \otimes \sigma_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0\end{array}\right). \tag{3.10}$$

[1] Actually, U_0 may be multiplied by a unitary block-diagonal matrix

$$U_0 \longrightarrow U_\Lambda = \left(\begin{array}{c|c} e^{i\lambda_1}\mathbb{I}_2 & 0 \\ \hline 0 & e^{i\lambda_2}\mathbb{I}_2\end{array}\right) \cdot U_0,$$

but the arbitrary phases λ_1 and λ_2 do not enter the game.

Now, any antisymmetric unitary matrix U may be represented as

$$U = VU_0V^T, \tag{3.11}$$

for some orthogonal matrix V. It shows that a general Breuer–Hall map φ_U^4 is unitary equivalent to φ_0^4

$$\varphi_U^4(X) = V\varphi_{U_0}^4(V^TXV)V^T, \tag{3.12}$$

and hence (after normalization) to the Robertson map

$$\widetilde{\varphi}_U^4(X) = (V\Gamma)\varphi_R(V^TXV)(V\Gamma)^T, \tag{3.13}$$

with $U_1 = V\Gamma$ and $U_2 = V$. □

Note, that for $V = \mathbb{I}_4$, one obtains

$$\widetilde{\varphi}_{U_0}^4(e_{ii}) = \varphi_R(e_{ii}), \tag{3.14}$$

$$\widetilde{\varphi}_{U_0}^4(e_{ij}) = -\varphi_R(e_{ij}), \qquad i \neq j. \tag{3.15}$$

Actually, one has

$$\widetilde{\varphi}_{U_0}^4(e_{ij}) = 0, \qquad i + j \in \{3, 7\}.$$

It was already shown by Robertson [19] that φ_R is indecomposable. However, it turns out that one may prove the following much stronger property:

Theorem 2. *Robertson map φ_R is atomic.*

Proof. To prove atomicity of φ_R one has to construct a PPT state $\rho \in (M_4 \otimes M_4)^+$ such that: (1) both ρ and its partial transpose ρ^τ are of Schmidt rank two and (2) entanglement of ρ is detected by the corresponding entanglement witness

$$W_R = (\mathbb{1} \otimes \varphi_R)P_4^+ = \sum_{i,j=1}^4 e_{ij} \otimes \varphi_R(e_{ij}).$$

One easily finds

$$W_R = \frac{1}{2}\left(\begin{array}{cccc|cccc|cccc|cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right), \tag{3.16}$$

where to maintain a more transparent form we replace all zeros by dots. Note, that W_R has single negative eigenvalue '-1', '0' (with multiplicity 10) and '+1' (with multiplicity 5). Consider now the following state constructed by Ha [33]:

$$\rho_{\mathrm{Ha}} = \frac{1}{7}\left(\begin{array}{cccc|cccc|cccc|cccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right). \tag{3.17}$$

It turns out [33] that ρ_{Ha} is PPT, and both ρ_{Ha} and $(\mathbb{1}\otimes\tau)\rho_{\mathrm{Ha}}$ have Schmidt rank 2. One easily finds

$$\mathrm{Tr}(W_R\rho_{\mathrm{Ha}}) = -1/14 < 0, \tag{3.18}$$

which proves atomicity of φ_R.[2] $\square$

Corrolary 1. *The Breuer–Hall map φ^4_U is atomic.*

Proof. Using the relation between $\widetilde{\varphi}^4_U$ and the Roberston map φ_R

$$\widetilde{\varphi}^4_U(X) = U_1\varphi_R(U_2^* X U_2)U_1^*, \tag{3.20}$$

let us compute $\mathrm{Tr}(\rho W_U)$, where

$$W^4_U = \big(\mathbb{1}\otimes\widetilde{\varphi}^4_U\big)P^+_4, \tag{3.21}$$

and ρ is an arbitrary state in $4\otimes 4$. One obtains

$$\mathrm{Tr}\big(\rho W^4_U\big) = \mathrm{Tr}\left(\rho\cdot\sum_{i,j=1}^{4} e_{ij}\otimes\widetilde{\varphi}^4_U(e_{ij})\right) = \mathrm{Tr}\left(\rho\cdot\sum_{i,j=1}^{4} e_{ij}\otimes U_1\varphi_R(U_2^* e_{ij}U_2)U_1^*\right).$$

[2] Note, that ρ_{Ha} is trivially extended from the following state in $3\otimes 3$:

$$\frac{1}{7}\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right), \tag{3.19}$$

which, therefore, provides an example of a bound entangled state.

Now, introducing $\widetilde{e}_i = U_2^* e_i$, one has

$$\begin{aligned}\mathrm{Tr}(\rho W_U) &= \mathrm{Tr}\left(\rho \cdot \sum_{i,j=1}^{4} U_2 \widetilde{e}_{ij} U_2^* \otimes U_1 \varphi_R(\widetilde{e}_{ij}) U_1^*\right)\\ &= \mathrm{Tr}(\rho \cdot (U_2 \otimes U_1)(\mathbb{1} \otimes \varphi_R) P_4^+ (U_2 \otimes U_1)^*)\\ &= \mathrm{Tr}((U_2 \otimes U_1)^* \rho (U_2 \otimes U_1) \cdot W_R). \end{aligned} \tag{3.22}$$

Hence, if ρ_{Ha} witnesses atomiticity of φ_R, then $(U_2 \otimes U_1)\rho_{\mathrm{Ha}}(U_2 \otimes U_1)^*$ witnesses atomiticity of φ_U^4. □

The above result may be immediately generalized as follows:

Corrolary 2. *If a positive map $\varphi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ is atomic, then $\widetilde{\varphi} : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ defined by*

$$\widetilde{\varphi}(X) := U_1 \varphi(U_2^* X U_2) U_1^*, \tag{3.23}$$

is atomic for arbitrary unitary operators U_1 and U_2 ($U_k : \mathcal{H}_k \longrightarrow \mathcal{H}_k; k = 1, 2$).

Theorem 3. *The Breuer–Hall map $\varphi_U^d : M_d \longrightarrow M_d$ with even d is atomic.*

Proof. Let Σ be a four-dimensional subspace in $\mathbb{C}^d$. It is clear that $U_\Sigma := U|_\Sigma$ gives rise to the Breuer–Hall map in four dimensions

$$\varphi_{U_\Sigma}^4 : \mathcal{B}(\Sigma) \longrightarrow \mathcal{B}(U(\Sigma)).$$

This map is atomic and hence it is witnessing by a 4×4 density matrix supported on Σ, such that ρ is PPT, Schmidt rank of ρ and its partial transposition equals 2, and such that $\mathrm{Tr}\big(\rho W_{U_\Sigma}^4\big) < 0$. Let us extend the 4×4 state ρ into the following $d \otimes d$ state:

$$\widehat{\rho}_{ij,kl} = \begin{cases} \rho_{ij,kl}, & i, j, k, l \leqslant 4 \\ 0 & \text{otherwise,} \end{cases} \tag{3.24}$$

where we take a basis $(e_1, \ldots, e_d)$ such that $e_1, \ldots, e_4 \in \Sigma$. It is clear that extended $\widehat{\rho}$ is PPT in $d \otimes d$ and Schmidt rank of $\widehat{\rho}$ and $(\mathbb{1} \otimes \tau)\widehat{\rho}$ equals again 2. Moreover

$$\mathrm{Tr}\big(\widehat{\rho} W_U^d\big) = \mathrm{Tr}\big(\rho W_{U_\Sigma}^4\big) < 0, \tag{3.25}$$

which proves atomicity of φ_U^d. □

Let us observe that d need not be even. Indeed, let $d \geqslant 4$ and let U be the antisymmetric unitary operator $U : \Sigma \longrightarrow \Sigma$, where Σ denotes an arbitrary even-dimensional subspace of $\mathbb{C}^d$. One extends U to an operator $\widehat{U}$ in $\mathbb{C}^d$ by

$$\widehat{U}(x, y) = (Ux, 0), \tag{3.26}$$

where $x \in \Sigma$ and $y \in \Sigma^\perp$, and hence, $\widehat{U}$ is still antisymmetric but no longer unitary in $\mathbb{C}^d$. Finally, let us define

$$\varphi_{\widehat{U}}^d(X) = \mathrm{Tr}(X)\mathbb{I}_d - X - \widehat{U} X^T \widehat{U}^*, \tag{3.27}$$

that is, it acts as the standard Breuer–Hall map on $\mathcal{B}(\Sigma)$ only. Note, that

$$\varphi_{\widehat{U}}^d(\mathbb{I}_d) = (d-2)\mathbb{I}_d + P^\perp, \tag{3.28}$$

where $P^\perp$ denotes a projector onto $\Sigma^\perp$. Therefore, the normalized map reads as follows:

$$\widetilde{\varphi}_{\widehat{U}}^d(X) = [(d-2)\mathbb{I}_d + P^\perp]^{-1/2} \cdot \varphi_{\widehat{U}}^d(X) \cdot [(d-2)\mathbb{I}_d + P^\perp]^{-1/2}, \tag{3.29}$$

and has much more complicated form than (3.2).

Theorem 4. *The formula (3.27) with arbitrary $d \geqslant 4$ and even-dimensional subspace Σ (with $\dim\Sigma \geqslant 4$) defines a positive atomic map.*

Proof. Let $d > \dim\Sigma = 2k \geqslant 4$. It is clear that

$$\varphi_U^{2k} := \varphi_{\widehat{U}}^d\big|_{\mathcal{B}(\Sigma)}, \tag{3.30}$$

defines the standard Breuer–Hall map in $\mathcal{B}(\Sigma)$. Now, due to theorem 3 the map φ_U^{2k} is atomic. If ρ is a $2k \otimes 2k$ state living in $\Sigma \otimes \Sigma$ witnessing atomicity of φ_U^{2k}, then trivially extended $\widehat{\rho}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ witnesses atomicity of $\varphi_{\widehat{U}}^d$. □

4. New classes of atomic maps

Now we are ready to propose a generalization of the class of positive maps considered by Hall [45]

$$\varphi(X) = \sum_{k<l}\sum_{m<n} c_{kl,mn} A_{kl} X^T A_{mn}^*, \tag{4.1}$$

where

$$A_{kl} = e_{kl} - e_{lk}, \tag{4.2}$$

with $c_{kl,mn}$ being a $d \times d$ Hermitian matrix. One example of such a map is a Breuer–Hall one

$$\varphi_U^d(X) = \mathrm{Tr}(X)\mathbb{I}_d - X - UX^TU^*, \tag{4.3}$$

which is shown to be atomic. The other example is provided by the well-known reduction map

$$R(X) = \mathrm{Tr}(X)\mathbb{I}_d - X. \tag{4.4}$$

This map is completely co-positive and hence decomposable. It is therefore clear that any convex combination

$$\phi_x^U(X) = x\varphi_U^d(X) + (1-x)R(X) = \mathrm{Tr}(X)\mathbb{I}_d - X - xUX^TU^*, \tag{4.5}$$

for $x \in [0, 1]$, defines a positive map from the class (4.1). Note, that if $\mathrm{rank}\, U = 2k < d$, then the matrix $[c_{kl,mn}]$ possesses a negative eigenvalue '$1 - xk$' for x satisfying

$$\frac{1}{k} < x \leqslant 1, \tag{4.6}$$

and hence $\phi_x^U(X)$ is indecomposable if (4.6) is satisfied. Finally, let us generalize (4.5) and consider a 2-parameter family

$$\chi_{x,y}^U(X) = \mathrm{Tr}(X)\mathbb{I}_d - yX - xUX^TU^*. \tag{4.7}$$

It is clear that for $y \in [0, 1]$ the above map is positive. Note, that $\chi_{0,0}^U(X) = \mathrm{Tr}(X)\mathbb{I}_d$ is completely positive whereas $\chi_{1,1}^U$ reproduces the Breuer–Hall map φ_U^d. Now, we are going to establish the range of $(x, y) \in [0, 1] \times [0, 1]$ for which $\chi_{x,y}^U$ is atomic.

Theorem 5. *A positive map $\chi_{x,y}^U$ is atomic if $x + y > 7/4$.*

Proof. Let us start with $d = 4$ and $\Sigma = \mathbb{C}^4$ and consider $\chi_{x,y}^U$ with $U = U_0$ defined in (3.10)

$$\chi_{x,y}^{U_0}(X) = \mathrm{Tr}(X)\mathbb{I}_4 - yX - xU_0X^TU_0^*. \tag{4.8}$$

Let $W^{U_0}_{x,y}$ be the corresponding entanglement witness

$$W^{U_0}_{x,y} = \left(\mathbb{1} \otimes \chi^{U_0}_{x,y}\right) P^+_4 =$$

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
1-x & \cdot & \cdot & \cdot & \cdot & y-x & \cdot & \cdot & \cdot & \cdot & -x & \cdot & \cdot & \cdot & \cdot & -x \\
\cdot & 1-y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1-y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
y-x & \cdot & \cdot & \cdot & \cdot & 1-x & \cdot & \cdot & \cdot & \cdot & -x & \cdot & \cdot & \cdot & \cdot & -x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -y & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-x & \cdot & \cdot & \cdot & \cdot & -x & \cdot & \cdot & \cdot & \cdot & 1-x & \cdot & \cdot & \cdot & \cdot & y-x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1-y & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -y & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1-y & \cdot \\
-x & \cdot & \cdot & \cdot & \cdot & -x & \cdot & \cdot & \cdot & \cdot & y-x & \cdot & \cdot & \cdot & \cdot & 1-x
\end{array}\right), \tag{4.9}$$

It is easy to show that

$$\mathrm{Tr}\big((\Gamma \rho_{\mathrm{Ha}} \Gamma^*) W^{U_0}_{x,y}\big) = \tfrac{2}{7}(7 - 4x - 4y), \tag{4.10}$$

where ρ_{Ha} is defined in (3.17). Hence, if $7 - 4(x + y) < 0$, then $\chi^{U_0}_{x,y}$ is atomic. Now, it is clear from the proofs of theorems 1 and 4 that the same result applies for arbitrary d and arbitrary U. □

Similarly, we may find a region in (x, y) square where $\chi^U_{x,y}$ is indecomposable. One has

Theorem 6. *A positive map $\chi^U_{x,y}$ is indecomposable if $x + y > 3/2$.*

Proof. Similarly, as in the proof of the previous theorem, one computes

$$\mathrm{Tr}\big((\Gamma \rho_{\mathrm{new}} \Gamma^*) W^{U_0}_{x,y}\big) = 3 - 2x - 2y, \tag{4.11}$$

where ρ_{new} is defined by

$$\rho_{\mathrm{new}} = \frac{1}{24}\left(\begin{array}{cccc|cccc|cccc|cccc}
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2
\end{array}\right), \tag{4.12}$$

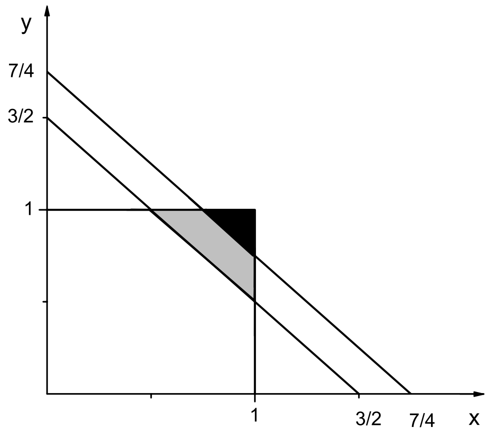

Figure 1. Regions of indecomposability (gray and black) and of atomicity (black).

and turns out to be PPT.[3] It is therefore clear that for $x + y > 3/2$, that map $\chi^{U_0}_{x,y}$ is indecomposable. Using the same techniques as in the proof of theorem 5 we prove that $x + y > 3/2$ guaranties indecomposability for arbitrary d and U. □

The regions of indecomposability ($x + y > 3/2$) and of atomicity ($x + y > 7/4$) are displayed in figure 1. We stress that these regions are derived by using specific states: ρ_{new} and ρ_{Ha}, respectively. It is interesting to look for other states which are 'more optimal' and enable us to enlarge these regions.

Finally, let us observe that the family $\chi^{U}_{x,y}$ may be further generalized as follows: consider a set of N antisymmetric unitary 4×4 matrices $\mathbf{U} = (U_1, \ldots, U_N)$ and let $\mathbf{x} = (x_1, \ldots, x_N)$ be a set of N non-negative numbers. Define the following map[4]

$$\Psi^{\mathbf{U}}_{\mathbf{x},y}(X) = \mathrm{Tr}(X)\mathbb{I}_4 - yX - \sum_{k=1}^{N} x_k U_k X^T U_k^*. \tag{4.15}$$

It is clear that if $U_1 = \cdots = U_N =: U$, then $\Psi^{\mathbf{U}}_{\mathbf{x},y} = \chi^{U}_{x,y}$ with $x := x_1 + \cdots + x_N$. Note, that for an arbitrary rank-1 projector P each $Q_k = U_k P U_k^*$ is again rank-1 projector orthogonal to P. However, for $U_k \neq U_l$ projectors Q_k and Q_l are no longer mutually orthogonal. Let us observe that for $x \leqslant 1$ the map $\Psi^{\mathbf{U}}_{\mathbf{x},y}$ is positive for arbitrary antisymmetric unitary $\mathbf{U}$ but the indecomposability/atomicity of $\Psi^{\mathbf{U}}_{\mathbf{x},y}$ depends upon $\mathbf{U}$ and it deserves further studies.

5. Conclusions

We provided a new large class of positive atomic maps in the matrix algebra M_d. These maps generalize a class of maps discussed recently by Breuer [46] and Hall [45]. The importance

[3] Actually, we originally constructed ρ_{new} to 'beat' (3.18). One finds

$$\mathrm{Tr}(W_R\rho_{\text{new}}) = -1/6, \tag{4.13}$$

which is 'much better' than $-1/14$. We conjecture, that ρ_{new} is 'optimal' in the following sense:

$$\min_{\rho\in\text{PPT}} \mathrm{Tr}(W_R\rho) = -1/6, \tag{4.14}$$

that is, ρ_{new} minimizes $\mathrm{Tr}(W_R\rho)$ among all PPT states.

[4] We thank Dr R Augusiak for his remarks.

of atomic maps follows from the fact that they may be used to detect the 'weakest' bound entanglement, that is, atomic maps can detect entangled states from $V_2 \cap V^2$. By duality, these maps provide a new class of atomic entangled witnesses. Note, that if φ is atomic and $(\mathbb{1} \otimes \varphi)\rho \ngeq 0$, then $\rho \in V_2 \cap V^2$ and hence ρ may be used as a test for atomicity of positive indecomposable maps. Since we know only few examples of quantum states belonging to $V_2 \cap V^2$ any new example of this kind is welcome. It is hoped that new maps provided in this paper find applications in the study of 'weakly' entangled PPT states. For example in recent papers [51] and [52] we constructed very general classes of PPT states in $d \otimes d$. It would be interesting to search for entangled states within these classes by applying our new family of indecomposable and atomic maps.

Acknowledgments

This work was partially supported by the Polish Ministry of Science and Higher Education grant no 3004/B/H03/2007/33. We thank referees for valuable remarks.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum computation and quantum information* (Cambridge: Cambridge University Press)
[2] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[3] Horodecki P 1997 *Phys. Lett.* A **232** 333
[4] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 8
[5] Terhal B M 2000 *Phys. Lett.* A **271** 319
[6] Jamiołkowski A 1972 *Rep. Math. Phys.* **3** 275
[7] Choi M-D 1975 *Linear Algebra Appl.* **10** 285
Choi M-D 1975 *Linear Algebra Appl.* **12** 95
[8] Stinespring W F 1955 *Proc. Am. Math. Soc.* **6** 211
[9] Størmer E 1963 *Acta Math.* **110** 233
[10] Størmer E 1965 *Trans. Am. Math. Soc.* **120** 438
[11] Størmer E 1974 *Lecture Notes in Physics* vol 29 (Berlin: Springer) pp 85–106
[12] Størmer E 1982 *Proc. Am. Math. Soc.* **86** 402
[13] Arverson W 1969 *Acta Math.* **123** 141
[14] Choi M-D 1980 *J. Operator Theory* **4** 271
[15] Woronowicz S L 1976 *Rep. Math. Phys.* **10** 165
[16] Woronowicz S L 1976 *Commun. Math. Phys.* **51** 243
[17] Robertson A G 1983 *Q. J. Math. Oxford* 2 **34** 87
[18] Robertson A G 1983 *Proc. R. Soc. Edinburh* A **94** 71
Robertson A G 1983 *Proc. R. Soc. Edinburh* A **34** 87
[19] Robertson A G 1983 *Math. Proc. Camb. Phil. Soc.* **94** 71
[20] Robertson A G 1985 *J. London Math. Soc.* 2 **32** 133
[21] Tang W-S 1986 *Linear Algebra Appl.* **79** 33
[22] Itoh T 1986 *Math. Japonica* **31** 607
[23] Takasaki T and Tomiyama J 1982 *Math. Japonica* **1** 129
[24] Tomiyama J 1987 *Contemp. Math.* **62** 357
[25] Tanahashi K and Tomiyama J 1988 *Can. Math. Bull.* **31** 308
[26] Osaka H 1991 *Linear Algebra Appl.* **153** 73
Osaka H 1993 *Linear Algebra Appl.* **186** 45
[27] Osaka H 1992 *Publ. RIMS Kyoto Univ.* **28** 747
[28] Cho S J, Kye S-H and Lee S G 1992 *Linear Algebra Appl.* **171** 213
[29] Kim H-J and Kye S-H 1994 *Bull. Lond. Math. Soc.* **26** 575
[30] Kye S-H 1997 *Math. Proc. Camb. Philos. Soc.* **122** 45
[31] Kye S-H 2003 *Linear Algebra Appl.* **362** 57
[32] Eom M-H and Kye S-H 2000 *Math. Scand.* **86** 130
[33] Ha K-C 1998 *Publ. RIMS Kyoto Univ.* **34** 591

[34] Ha K-C 2002 *Linear Algebra Appl.* **348** 105
Ha K-C 2003 *Linear Algebra Appl.* **359** 277
[35] Ha K-C, Kye S-H and Park Y S 2003 *Phys. Lett.* A **313** 163
[36] Ha K-C and Kye S-H 2004 *Phys. Lett.* A **325** 315
[37] Ha K-C and Kye S-H 2005 *J. Phys. A: Math. Gen.* **38** 9039
[38] Terhal B M 2001 *Linear Algebra Appl.* **323** 61
[39] Majewski W A and Marcinek M 2001 *J. Phys. A: Math. Gen.* **34** 5836
[40] Kossakowski A 2003 *Open Syst. Inform. Dyn.* **10** 1
[41] Kimura G and Kossakowski A 2004 *Open Syst. Inform. Dyn.* **11** 1
Kimura G and Kossakowski A 2004 *Open Syst. Inform. Dyn.* **11** 343
[42] Benatti F, Floreanini R and Piani M 2004 *Phys. Lett.* A **326** 187
[43] Piani M 2006 *Phys. Rev.* A **73** 012345
[44] Chruściński D and Kossakowski A 2007 *Open Syst. Inform. Dyn.* **14** 275
[45] Hall W 2006 *J. Phys. A: Math. Gen.* **39** 14119
[46] Breuer H-P 2006 *Phys. Rev. Lett.* **97** 0805001
[47] Breuer H-P 2005 *Phys. Rev.* A **71** 062330
Breuer H-P 2005 *J. Phys. A: Math. Gen.* **38** 9019
[48] Augusiak R and Stasińska J 2007 *Phys. Lett.* A **363** 182
[49] Lewenstein M, Kraus B, Cirac J I and Horodecki P 2000 *Phys. Rev.* A **62** 052310
[50] Terhal B and Horodecki P 2000 *Phys. Rev.* A **61** 040301
Sanpera A, Bruss D and Lewenstein M 2001 *Phys. Rev.* A **63** 050301(R)
[51] Chruściński D and Kossakowski A 2006 *Phys. Rev.* A **74** 022308
[52] Chruściński D and Kossakowski A 2007 *Phys. Rev.* A **76** 032308